UNIVERSITY OF ILLINOIS ENGINEERING EXPERIMENT STATION

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ALMOST SINUSOIDAL OSCILLATIONS IN NONLINEAR SYSTEMS
Part II: Synchronization

J. S. Schaffner

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Part II: Synchronization

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. INTRODUCTION

In the absence of the external voltage v_s the circuit of ig. 1.1 is a tuned-plate oscillator. It consists essentially f a resonant circuit and an electron tube. This electron tube as a double function. It feeds power into the resonant circuit n order to compensate for the losses in the passive elements and ecause of its nonlinearity, it limits the amplitude of oscilla- ion. The frequency of oscillation in the first approximation s $\omega_o = 1/\sqrt{LC}$.

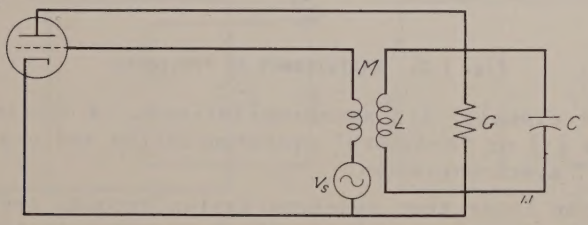

Fig. 1.1. Tuned-Plate Oscillator with External Sinusoidal Synchronizing Voltage

If the external voltage v_s is not zero and if its frequency is aried over a wide range, then the frequency ω of the free oscil- ation does not remain constant but behaves as indicated in Fig. 1.2. If the frequency ω_s of the external voltage is varied and ecomes approximately equal to $(p/q)\omega_o$, where p and q are "small" ntegers, then the frequency of oscillation suddenly changes from $_o$ to $(q/p)\omega_s$. This phenomenon is called locking, synchroniza- ion, or entrainment of frequency. The external voltage will synchronize the free oscillation over a frequency range $\Delta\omega$ (Fig. 1.2). The expression $(\Delta\omega/\omega)_o = \Delta\omega/\omega_o$ is defined as the bandwidth of synchronization."

For synchronization the relationship between ω and ω_s is

$$p\omega = q\omega_s \qquad (1.1)$$

For particular values of p and q the term "$p:q$ synchronization"

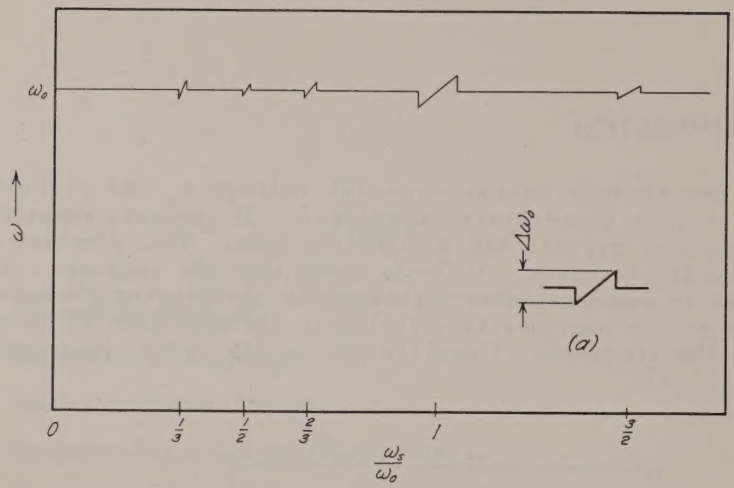

Fig. 1.2. Entrainment of Frequency

is used (for example; 1:3 synchronization). A distinction made between 1:1 or "ordinary" synchronization and $p:q$ $(p \neq q)$ "subharmonic" synchronization.

It is shown later that synchronization becomes very weak the sum $(p+q)$ increases. Any small fluctuation in the circ then puts the system out of synchronism. If $(p+q)$ is sufficie ly large, then synchronization can be neglected altogether. considerable number of papers have been written on synchroni tion.[3-14]* Unfortunately, most do not yield comprehensive sults. It is hoped that this bulletin may fill some gaps.

*Parenthesized, superscript numerals refer to correspondingly numbered entrie the Selected Bibliography.

II. OUTLINE OF THE METHOD

1. Definitions of Terms

The circuit of Fig. 1.1 shows only one of the many ways by which an oscillator can be synchronized. Instead of being limited to this special case, it is desirable that the results of this discussion be applicable to a great number of oscillators. The circuit to be discussed must therefore be rather general. Two such general oscillators are shown in Fig. 2.1. They consist

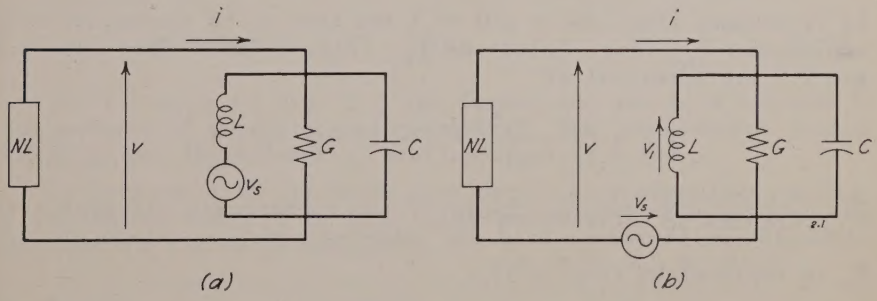

Fig. 2.1. General Oscillatory Circuits

of a resonant circuit, a synchronizing voltage v_s and a "nonlinear" element. This nonlinear element has much the same function as the electron tube in Fig. 1.1; it supplies power to the passive elements of the circuit and limits the amplitude of oscillation. The discussion is limited to nonlinear elements with a current-voltage relationship that can be expressed by the rapidly converging power series

$$i = \alpha v + \beta v^2 + \gamma v^3 + \delta v^4 \qquad (2.1)$$

In this bulletin it is always assumed that the synchronizing voltage v_s is sinusoidal. It may be applied to the oscillator in two essentially different ways. In Fig. 2.1a, v_s is inserted in the resonant circuit; in Fig. 2.1b, it is placed in series with the nonlinear element. The first case is called "internal," the second "external" synchronization.

For 1:1 synchronization, internal and external synchronization are discussed separately since one case cannot be easily reduced

to the other. For subharmonic synchronization, however, the discussion is limited to external synchronization. As was pointed out to the author by Dr. Giuseppe Francini, the result can then be applied easily to internal synchronization as follows: The differential equation corresponding to the circuit of Fig. 2.1 is

$$C\frac{dv}{dt} + Gv + (\alpha v + \beta v^2 + \gamma v^3 + \ldots) + \frac{1}{L}\int v\,dt = \frac{1}{L}\int v_s\,dt \qquad (2.2)$$

and to that of Fig. 2.1b is

$$C\frac{dv}{dt} + Gv + (\alpha v + \beta v^2 + \gamma v^3 + \ldots) + \frac{1}{L}\int v\,dt = C\frac{dv_s}{dt} + Gv_s + \frac{1}{L}\int v_s\,dt \qquad (2.3)$$

It is assumed that $G/\omega C = 1/Q << 1$ and that v_s is sinusoidal with amplitudes V_{sa} (Fig. 2.1a) and V_{sb} (Fig. 2.1b). Then Eqs. 2.2 and 2.3 are identical if

$$V_{sa} = \left(1 - \frac{\omega_s^2}{\omega^2}\right) V_{sb} \cong (1 - p^2/q^2)\, V_{sb} \qquad (2.4)$$

The results obtained in Chapter V for subharmonic external synchronization therefore hold for internal synchronization also if V_s is replaced by $(1-p^2/q^2)V_s$.

2. The Linearization of the Oscillator

All oscillators contain nonlinear elements. These nonlinearities make the mathematical treatment difficult, since they do not allow solutions in closed form. It is therefore necessary to use approximation methods. The amount of calculation becomes prohibitive for higher approximations; it is therefore essential that the first approximation be very good. Such a reasonably accurate first approximation can be found if physical considerations (high Q) indicate that the oscillations are "almost sinusoidal."

For example, for small v_s the voltage v in Fig. 2.1a is almost sinusoidal and is given by

$$v \cong V\cos(\omega t + \phi)$$

The voltage v_1 in Fig. 2.1b is

$$v_1 \cong V\cos(\omega t + \phi) \qquad (2.5)$$

If the oscillator is synchronized, then $\omega = (q/p)\omega_s$, and the current i passing through the nonlinear element will contain more higher harmonics than the voltage across it. The current can be

calculated from Eqs. 2.1 and 2.5. It has the general form

$$i = I_1 \cos(\omega t+\phi) + I_2 \sin(\omega t+\phi) + \text{components at frequencies other than } \omega \qquad (2.6)$$

This current, passing through the resonant circuit, influences the oscillation somewhat. The components at frequencies other than ω cause forced oscillations at these frequencies. The amplitudes of these forced oscillations are necessarily small if the Q of the resonant circuit is high, and they can therefore be neglected. As is shown later, the components at the frequency ω will, however, cause a small change in amplitude V and phase ϕ of the oscillation.

If all components of the current except those at frequency ω are neglected, then the circuits of Fig. 2.1 can be replaced by those of Fig. 2.2 where

$$G_e = \frac{I_1}{V} \; ; \; C_e = -\frac{I_2}{\omega V} \qquad (2.7)$$

The circuits of Fig. 2.2 are linear and can be discussed by the methods of linear circuit analysis. The "equivalent" conductance G_e and capacitance C_e are functions of V and ϕ.

In Section 4 it is shown that for 1:1 synchronization Fig. 2.2a can be reduced to Fig. 2.2b so that it is sufficient to discuss the latter circuit only.

3. Equilibrium and Stability

The frequency of oscillation of the circuit of Fig. 2.2b is

$$\omega_1 = \frac{1}{\sqrt{L(C+C_e)}} \cong \omega_o\left(1 - \frac{C_e}{2C}\right) \qquad (2.8)$$

where $\omega_o = 1/\sqrt{LC}$. It is assumed that

$$v = V \cos\left(\frac{q}{p}\omega_s t+\phi\right)$$

Should ω_1 differ from $(q/p)\omega_s$, the phase angle will change slowly

$$\omega_1 = \frac{q}{p}\omega_s + \frac{d\phi}{dt}$$

or

$$\frac{d\phi}{dt} = \Delta\omega - \omega\frac{C_e}{2C} \qquad (2.8)$$

where $\Delta\omega = \omega_o - (q/p)\omega_s$.

Similarly for V the energy stored in the resonant circuit, $CV^2/2$, will change slowly since part of it is dissipated in the conductances G_e and G; that is,

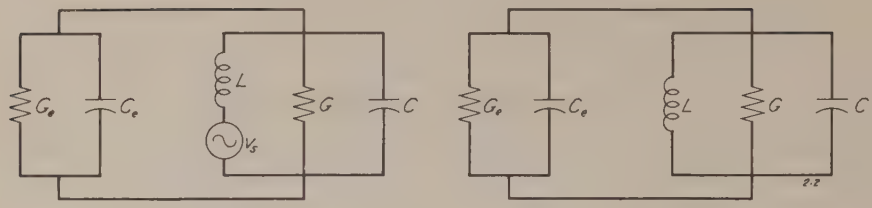

Fig. 2.2. Equivalent Linearized Oscillatory Circuits

$$\frac{d}{dt}\left(\frac{CV^2}{2}\right) = -\frac{V^2}{2}(G+G_e)$$

or

$$\frac{dV}{dt} = -\frac{V}{2C}(G+G_e) \tag{2.10}$$

For steady-state oscillations, both V and ϕ must remain constant. Therefore

$$\begin{aligned} G + G_e &= 0 \\ \Delta\omega - \frac{C_e}{2C} &= 0 \end{aligned} \tag{2.11}$$

A steady-state oscillation is stable if the oscillator will resume its original amplitude and phase after a small disturbance. In order to discuss the behavior of the system in the neighborhood of the steady-state oscillations, it is necessary to expand Eqs. 2.9 and 2.10 around the amplitude V_o and the phase ϕ_o corresponding to steady-state oscillations. Defining

$$\begin{aligned} \delta V &= V - V_o \\ \delta\phi &= \phi - \phi_o \end{aligned}$$

the first terms of the expansion are

$$\begin{aligned} \frac{d(\delta V)}{dt} &= -\frac{V_o}{2C}\left[\frac{\partial G_e}{\partial V}\delta V + \frac{\partial G_e}{\partial \phi}\delta\phi\right] \\ \frac{d(\delta\phi)}{dt} &= -\frac{\omega}{2C}\left[\frac{\partial C_e}{\partial V}\delta V + \frac{\partial C_e}{\partial \phi}\delta\phi\right] \end{aligned} \tag{2.12}$$

It is assumed that δV and $\delta\phi$ are small and that therefore the higher terms of this expansion may be neglected.

Equation 2.12 is a set of two simultaneous linear differential equations. It can be solved by standard methods. The variables δV and $\delta\phi$ will approach zero from any initial conditions in the neighborhood of the steady-state oscillations if both the roots κ_1 and κ_2 of Eq. 2.13 have negative real parts.

$$\begin{vmatrix} \frac{V}{2C}\frac{\partial G_e}{\partial V} + \kappa & \frac{V}{2C}\frac{\partial G_e}{\partial \phi} \\ \frac{\omega}{2C}\frac{\partial C_e}{\partial V} & \frac{\omega}{2C}\frac{\partial C_e}{\partial \phi} + \kappa \end{vmatrix} = 0 \qquad (2.13)$$

Necessary and sufficient conditions for this are that

$$\begin{aligned} V\frac{\partial G_e}{\partial V} + \omega\frac{\partial C_e}{\partial \phi} &> 0 \\ \frac{\partial C_e}{\partial \phi}\frac{\partial G_e}{\partial V} - \frac{\partial C_e}{\partial V}\frac{\partial G_e}{\partial \phi} &> 0 \end{aligned} \qquad (2.14)$$

Systems satisfying Eqs. 2.11 and Ineqs. 2.14 will produce stable steady-state oscillations

III. ORDINARY SYNCHRONIZATION

4. Internal 1:1 Synchronization

In Fig. 2.1a, the voltage V_s is inserted in the resona circuit in series with the inductance L. If the voltage v_s = cos $\omega_s t$ synchronizes the oscillation, then the voltage v acro the nonlinear element is

$$v = V \cos(\omega_s t + \phi) \tag{3.1}$$

In this Chapter the subscript s of ω_s is dropped, since for l synchronization $\omega = \omega_s$.

It is assumed that the voltage v and the current i passi through the nonlinear network are related by

$$i = \alpha v + \beta v^2 + \gamma v^3 \tag{3.2}$$

where $\alpha < 0$, $\gamma > 0$. This is a good approximation for mo oscillators. It can then be shown that the equivalent impedan of Fig. 2.2a consists of a conductance G_e only (Fig. 3.1)

$$G_e = \alpha + \frac{3}{4}\gamma V^2 \tag{3.3}$$

If no synchronizing voltage is present ($v_s \equiv 0$), the conditi for steady-state oscillation is

$$G + G_e = 0$$

or

$$G + \alpha + \frac{3}{4}\gamma V^2 = 0 \tag{3.4}$$

The value of V satisfying this equation is

$$V_o^2 = -\frac{4}{3}\frac{G+\alpha}{\gamma} \tag{3.5}$$

The discussion in this section is limited to oscillators f which $V_o^2 > 0$.

In Fig. 3.1 the current passing through the inductance L is

$$i_L = \frac{1}{\omega L}\left[V \sin(\omega t + \phi) - V_s \sin \omega t\right] \tag{3.6}$$

The same current would flow through the network consisting of G_1 and C_1 due to the voltage $v = V \sin(\omega t + \phi)$ alone. The valu for G_1 and C_1 are

$$G_1 = \frac{\omega C V_s}{V} \sin \phi$$
$$C_1 = \frac{C V_s}{V} \cos \phi \qquad (3.7)$$

Therefore in the presence of a synchronizing voltage steady-state oscillations occur if

$$G + G_1 + G_e = 0$$
$$\frac{\Delta\omega}{\omega} - \frac{C_1}{2C} = 0 \qquad (3.8)$$

where $\Delta\omega = 1/\sqrt{LC} - \omega$. In terms of V and ϕ, these equations are

$$G + \alpha + \frac{3}{4}\gamma V^2 + \omega C \frac{V_s}{V} \sin \phi = 0$$
$$\frac{\Delta\omega}{\omega} - \frac{V_s}{2V} \cos \phi = 0 \qquad (3.9)$$

The conditions for stability are

$$V\frac{\partial}{\partial V}(G_e + G_1) + \frac{\partial C_1}{\partial \phi} = \frac{3}{2}\gamma V^2 - 2\omega C \frac{V_s}{V} \sin \phi$$
$$= \frac{3}{2}\gamma (2V^2 - V_o^2) > 0$$
$$\frac{\partial C_1}{\partial \phi}\frac{\partial}{\partial V}(G_e + G_1) - \frac{\partial C_1}{\partial V}\frac{\partial}{\partial \phi}(G_e + G_1) \qquad (3.10)$$
$$= -\frac{3}{2}\gamma C V_s \sin \phi + \omega C^2 \frac{V_s^2}{V^3} > 0$$

After rearranging the terms, the conditions for stability are

$$2V^2 > V_o^2$$
$$\sin \phi < \frac{2\omega C}{3\gamma}\frac{V_s}{V^3} \qquad (3.11)$$

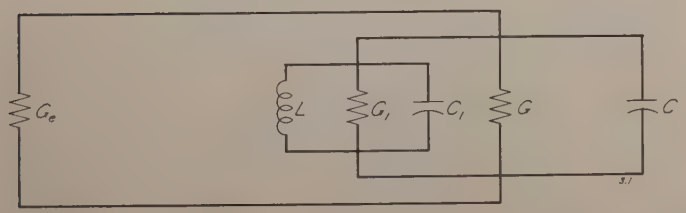

Fig. 3.1. Equivalent Linearized Circuit for Internal 1:1 Synchronization

This last condition is satisfied automatically for sin $\phi < 0$. F
sin $\phi > 0$, it can be simplified to

$$\sin^2 \phi < \frac{1}{2}\left(\frac{V_o^2}{V^2} - 1\right) \tag{3.12}$$

The equations for stability and equilibrium can be simplif
somewhat through the introduction of dimensionless parameters

$$X = \frac{V}{V_o}, \quad X_s = \frac{V_s}{V_o}, \quad A = -\frac{G + \alpha}{2\omega C} \tag{3.13}$$

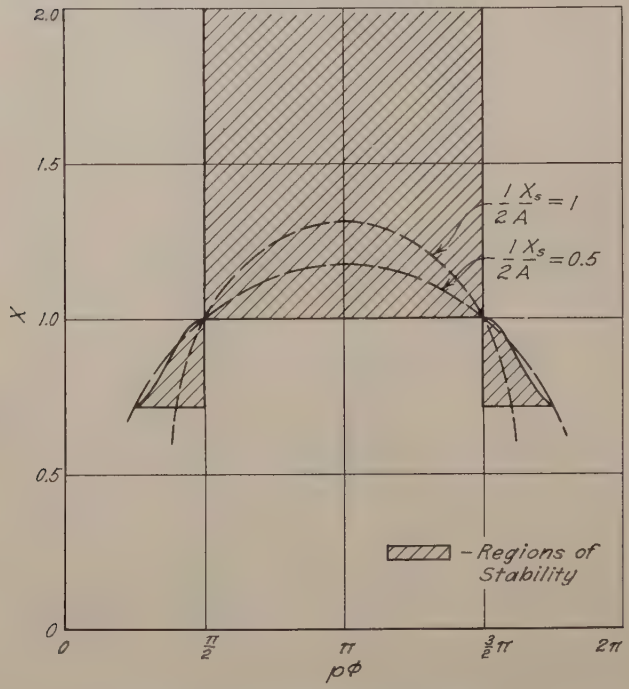

Fig. 3.2. Regions of Stability and Variation of X as a Function of $p\phi$ for $p = 1$, $q = 1$

Making these substitutions they become, for equilibrium,

$$\begin{aligned} A(X^2-1) + \frac{1}{2}\frac{X_s}{X}\sin\phi &= 0 \\ \frac{\Delta\omega}{\omega} - \frac{1}{2}\frac{X_s}{X}\cos\phi &= 0 \end{aligned} \tag{3.14}$$

and for stability

$$X^2 > \frac{1}{2}$$

and either

$$\sin \phi < 0$$

or if $\sin \phi > 0$

$$\sin^2\phi < \frac{1}{2}\left(\frac{1}{X^2} - 1\right)$$

The phase angle ϕ may be eliminated from these equations so that those for equilibrium reduce to

$$[A^2X^2(X^2-1)^2 + X^2(\frac{\Delta\omega}{\omega})^2] = \frac{1}{4}X_s^2 \qquad (3.16)$$

and those for stability to

$$(\frac{\Delta\omega}{\omega})^2 + (X^2 - 1)(3X^2 - 1) > 0 \qquad (3.17)$$

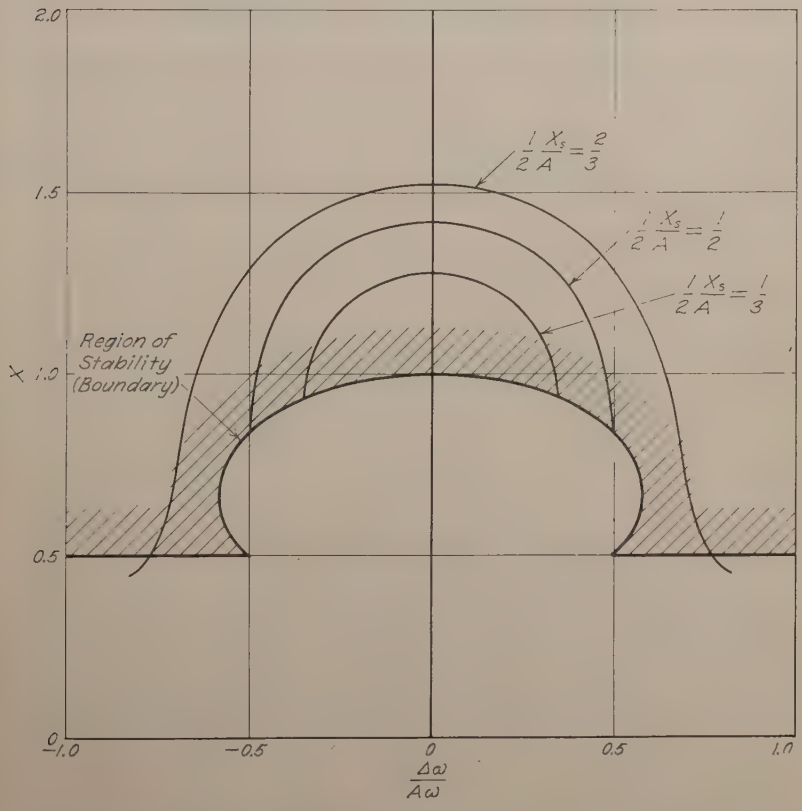

Fig. 3.3. Region of Stability and Variation of X as a Function of $\Delta\omega/A\omega$ for $p = 1$, $q = 1$

The regions of stability may be represented in a plane with X as the ordinate and either ϕ or $\frac{\Delta\omega}{A\omega}$ as the abscissa. (Figs. 3.2 and 3.3). These figures also indicate the variation of X with $\frac{\Delta\omega}{A\omega}$ and ϕ, respectively. As stated before, these figures have been discussed extensively in the literature.

5. External 1:1 Synchronization

In Fig. 2.1b, it can be assumed that the voltage v across the resonant circuit is approximately sinusoidal; that is

$$v_1 \cong V \cos(\omega t+\phi) \tag{3.18}$$

If the oscillator is synchronized by the external voltage, the frequency of oscillation is ω_s and as in Section 4 the subscript of ω_s is dropped.

The voltage across the nonlinear element is

$$\begin{aligned} v &= V \cos(\omega t+\phi) + V_s \cos \omega t \\ &= (V + V_s \cos \phi) \cos(\omega t+\phi) + V_s \sin \phi \sin(\omega t+\phi) \end{aligned} \tag{3.19}$$

It is again assumed that the current i and the voltage v are related by

$$i = \alpha v + \beta v^2 + \gamma v^3 \tag{3.2}$$

where $\alpha < 0$, $\gamma > 0$. Using Eqs. 3.19 and 3.2, the current i can be expressed as a function of time

$$\begin{aligned} i = {} & I_1 \cos(\omega t+\phi) + I_2 \sin(\omega t+\phi) \\ & + \text{components at frequency } 3\omega \end{aligned} \tag{3.20}$$

The circuit of Fig. 2.1b can then be replaced by that of Fig. 2.2b where

$$G_e = \frac{I_1}{V}, \quad C_e = -\frac{I_2}{\omega V} \tag{2.7}$$

The values of G_e and C_e are

$$\begin{aligned} G_e &= (1 + \frac{V_s}{V} \cos \phi) \left[\alpha + \frac{3}{4}\gamma(V^2 + 2VV_s \cos \phi + V_s^2)\right] \\ C_e &= -\frac{V_s}{\omega V} \sin \phi \left[\alpha + \frac{3}{4}\gamma(V^2 + 2VV_s \cos \phi + V_s^2)\right] \end{aligned} \tag{3.21}$$

If $V_s \ll V$, these equations can be simplified considerably to

$$\begin{aligned} G_e &= \alpha + \frac{3}{4}\gamma V^2 \\ C_e &= -\frac{V_s}{\omega V} \sin \phi \left(\alpha + \frac{3}{4}\gamma V^2\right) = -\frac{V_s}{\omega V} \sin \phi \, G_e \end{aligned} \tag{3.22}$$

As before the conditions for steady-state oscillations are

$$G + G_e = 0$$

$$\frac{\Delta\omega}{\omega} - \frac{C_e}{2C} = 0 \tag{3.23}$$

where $\Delta\omega = 1/\sqrt{LC} - \omega$. In terms of V and ϕ, Eqs. 3.23 are

$$G + \alpha + \frac{3}{4}\gamma V^2 = 0$$

$$\frac{\Delta\omega}{\omega} - \frac{V_s}{2\omega V} \sin\phi\, G_e = 0 \tag{3.24}$$

The conditions for stability are

$$V\frac{\partial G_e}{\partial V} + \omega\frac{\partial C_e}{\partial \phi} > 0$$

$$\frac{\partial G_e}{\partial V}\frac{\partial C_e}{\partial \phi} - \frac{\partial G_e}{\partial \phi}\frac{\partial C_e}{\partial V} > 0 \tag{3.25}$$

Since $\frac{\partial G_e}{\partial \phi} = 0$, these inequalities are satisfied if

$$\frac{\partial G_e}{\partial V} > 0, \quad \frac{\partial C_e}{\partial \phi} > 0 \tag{3.26}$$

or if

$$\gamma > 0 \quad \cos\phi > 0 \tag{3.27}$$

The permissible variation in ϕ is, therefore,

$$-\frac{\pi}{2} < \phi < \frac{\pi}{2}$$

The variation in frequency corresponding to this is

$$\left(\frac{\Delta\omega}{\omega}\right)_o = \frac{V_s}{V}\frac{G}{\omega C} = \frac{V_s}{V}\frac{1}{Q} \tag{3.28}$$

where $Q = \omega C/G$ is the quality factor of the resonant circuit. Thus the maximum variation in frequency is proportional to the ratio of amplitudes of the external synchronizing voltage to the internal voltage. Consequently if the external voltage increases, the bandwidth of synchronization increases likewise. In addition the maximum frequency variation is inversely proportional to the Q of the circuit. Therefore for small Q, the frequency can be entrained over a large band; but for high Q, the bandwidth of synchronization is relatively small.[7]

IV. SUBHARMONIC SYNCHRONIZATION

For external subharmonic synchronization the voltage v_1 across the resonant circuit of Fig. 2.1 is again assumed to be approximately sinusoidal

$$v_1 \cong V \cos(\omega t+\phi) \tag{3.17}$$

where ω is approximately $1/\sqrt{LC}$. For synchronization, ω and the frequency ω_s of the external voltage v_s are related by Eq. 1.1. The voltage across the nonlinear network is then

$$v = V \cos(\omega t+\phi) + V_s \cos \omega_s t \tag{4.1}$$

It is assumed that this voltage and the current i passing through the nonlinear network are related by a rapidly convergent power series

$$i = \alpha v + \beta v^2 + \gamma v^3 + \delta v^4 + \ldots \tag{2.1}$$

If this expression is combined with Eq. 4.1, then the current can be expressed as a function time

$$i = I_1 \cos(\omega t+\phi) + I_2 \sin(\omega t+\phi) + \text{components at frequencies other than } \omega \tag{2.6}$$

The circuit of Fig. 2.1b can then be replaced by that of Fig. 2.2b where again

$$G_e = \frac{I_1}{V} \qquad C_e = -\frac{I_2}{\omega V} \tag{2.7}$$

Methods for calculating G_e and C_e have already been discussed in University of Illinois Engineering Experiment Station Bulletin 395, Chapter II.[15]

6. Oscillations in the Absence of Synchronism

For some frequencies the external voltage will not synchronize the oscillator. No relation $p\omega = q\omega_s$ will then be preserved over any appreciable length of time and it can be assumed that ω/ω_s is irrational. The equivalent impedances for ω/ω_s irrational have been calculated in Bulletin 395, Chapter II.[15] They are

$$G_e' = \alpha + \frac{3}{4}\gamma(V^2+2V_s^2) + \ldots$$
$$C_e' = 0 \tag{4.2}$$

The prime indicates that no synchronism is present. To facilitate further calculations, it is assumed that neglecting all terms of G_e' other than the first two will not lead to appreciable errors. This is true for most oscillators.

Steady-state oscillations are possible if $\alpha < 0$, $\gamma > 0$ and

$$G + G_e' = 0 \tag{4.3}$$

or

$$G + \alpha + \frac{3}{4}\gamma(V^2 + 2V_s) = 0 \tag{4.4}$$

The value of V that satisfies this equation is

$$V_o^2 = -\frac{4}{3}\frac{G+\alpha}{\gamma} - 2V_s^2 \tag{4.5}$$

V_o^2 has a physical significance only if it is positive, otherwise it is just a parameter satisfying Eq. 4.5. If $V_o^2 > 0$, then the oscillations of the system are called "free" oscillations; if $V_o^2 < 0$, "forced" oscillations.

It is desirable that Eq. 2.1 be changed to a form corresponding to Eq. 11.1 of Bulletin 395.[15]

$$i = I_o \sum_1^\infty a_\lambda x^\lambda \tag{4.6}$$

where the a_λ and x are dimensionless. This can be done by defining

$$x = \frac{v}{|V_o|}, \quad a_1 = \frac{\alpha}{|V_o|}, \quad a_2 = \frac{\beta}{|V_o|^2}, \quad a_3 = \frac{\gamma}{|V_o|^3}, \quad \cdots \tag{4.7}$$

Similarly, dimensionless variables corresponding to V and v_s are defined as

$$X = \frac{V}{|V_o|}, \quad X_o = \frac{V_o}{|V_o|}, \quad X_s = \frac{V_s}{|V_o|}, \quad G_o = \frac{I_o}{|V_o|} \tag{4.8}$$

for free oscillations $X_o^2 = 1$; for forced oscillations $X_o^2 = -1$.

An expression that will be helpful in the next section is

$$G + G_e' = G + \alpha + \frac{3}{4}\gamma(V^2 + 2V_s^2) \tag{4.9}$$

Combining this with Eq. 4.5 and making the substitutions of Eq. 4.8

$$G + G_e' = \frac{3}{4}a_3 G_o(X^2 - X_o^2) \tag{4.10}$$

7. Synchronized Oscillations

Locking phenomena occur if the two frequencies ω and ω_s are related by an equation

$$p\omega = q\omega_s \tag{1.1}$$

where p and q are small integers. The special case $p = q$ is not considered in this chapter.

Some of the combination frequencies ($n\omega \pm m\omega_s$) are now identical with ω; as a consequence I_1 and I_2 and therefore G_e and C_e may contain additional terms. In fact these terms are necessary for the mechanism of synchronization. A double-prime is used to indicate such additional terms. For synchronization the equivalent impedances are

$$G_e = G_e' + C_e'' \qquad C_e = C_e'' \tag{4.11}$$

It is assumed that the series $\sum_1^\infty a_\lambda x^\lambda$ converges rapidly. It is then sufficient to consider only the first term that contributes to G'' and C''. As shown in Bulletin 395, Eq. 10.19,[15] this term is

$$I_o a_\lambda x^\lambda \qquad \lambda = p+q-1 \tag{4.12}$$

and the corresponding G_e'' and C_e'' are (Bulletin 395, Eqs. 11.4 and 11.5)

$$\begin{aligned} G_e'' &= \frac{a_\lambda G_o}{2^{\lambda-1}} \binom{\lambda}{q} X^{p-2} X_s^q \cos p\phi \\ C'' &= \frac{a_\lambda G_o}{2^{\lambda-1}\omega} \binom{\lambda}{q} X^{p-2} X_s^q \sin p\phi \end{aligned} \tag{4.13}$$

These can be simplified by introducing a coefficient

$$\eta = \frac{a_\lambda}{2^{\lambda-1}} \binom{\lambda}{q}$$

G_e'' and C_e'' are then

$$\begin{aligned} G_e'' &= \eta G_o X^{p-2} X_s^q \cos p\phi \\ C_e'' &= \eta \frac{G_o}{\omega} X^{p-2} X_s^q \sin p\phi \end{aligned} \tag{4.14}$$

Some values for η are presented in Table I. Equations 4.14 correspond to a first-order approximation. As stated above, none of the terms $I_o a_\lambda x^\lambda$ where $\lambda = (p+q-1)$ will contribute to G_e'' and C_e''. This does not hold if approximations higher than the first are considered.

For example, for $5\omega = \omega_s$ ($p = 5$, $q = 1$) the contribution of $I_o a_3 x^3$ towards G_e'' and C_e'' is for a second-order approximation

$$\begin{aligned} G_e'' &= \frac{27}{128} a_3^2 \frac{G_o^2}{\omega C} X^3 X_s \cos 5\phi \\ C_e'' &= \frac{27}{128} a_3^2 \frac{G_o^2}{\omega^2 C} X^3 X_s \sin 5\phi \end{aligned} \tag{4.15}$$

TABLE I
Values of η

p	q				
	1	2	3	4	5
1	...	1/2 a_2	1/4 a_3	1/8 a_4	1/16 a_5
2	a_2	...	1/2 a_4	...	3/16 a_6
3	3/4 a_3	3/4 a_4	...	15/32 a_6	21/64 a_7
4	1/2 a_4	...	5/8 a_6	...	7/16 a_8
5	5/16 a_5	15/32 a_6	35/64 a_7	35/64 a_8	...

The contribution due to a first-order approximation would be zero since $\lambda = 3$ is smaller than $(p+q-1) = 5$. In most cases the terms of Eq. 4.15 are small compared with those corresponding to $I_o a_5 x^5$ and a first-order approximation, particularly if the circuit has a large Q ($G/\omega C$ small). They can therefore be neglected.

As a result of Eqs. 4.14 the total equivalent impedances are then

$$G_e = G_e' + G_e'' = G_o \left[a_1 + \frac{3}{4}a_3(X^2+2X_s^2) + \eta X^{p-2} X_s^q \cos p\phi \right.$$

$$C_e = C_e'' \qquad = -\eta \frac{G_o}{\omega} X^{p-2} X_s^q \sin p\phi \tag{4.16}$$

As shown in Chapter II, steady-state oscillations can occur if

$$G + G_e = 0$$

$$\frac{\Delta\omega}{\omega} - \frac{C_e}{2C} = 0 \tag{2.11}$$

where

$$\frac{\Delta\omega}{\omega} = \frac{1}{\sqrt{LC}} - \frac{q}{p}\omega_s.$$

In terms of X and ϕ, these equations are

$$X^2 - X_o^2 + \frac{4}{3}\frac{\eta}{a_3} X^{p-2} X_s^q \cos p\phi = 0 \tag{4.17}$$

$$\frac{\Delta\omega}{\omega} + \eta \frac{G_o}{2\omega C} X^{p-2} X_s^q \sin p\phi = 0 \tag{4.18}$$

If the phase angle is eliminated, then these two equations combine into

$$\left[\frac{3}{4} a_3 (X^2 - X_o^2)\right]^2 + \left[2\frac{\Delta\omega}{\omega}\frac{\omega C}{G_o}\right]^2 = \left[\eta X^{p-2} X_s^q\right]^2 \tag{4.19}$$

Equations 4.18 and 4.19 permit a representation of X as a function of either $p\phi$ or $\Delta\omega/\omega$. For the purpose of this bulletin, $p\phi$ is the more adequate independent variable.

The conditions for stable steady-state oscillations are

$$X \frac{\partial G_e}{\partial X} + \omega \frac{\partial C_e}{\partial \phi} > 0$$
$$\frac{\partial C_e}{\partial \phi} \frac{\partial G_e}{\partial X} + \frac{\partial C_e}{\partial X} \frac{\partial G_e}{\partial \phi} > 0 \qquad (4.20)$$

The first of these two inequalities is, in terms of X and ϕ,

$$X\frac{\partial G_e}{\partial X} + \omega \frac{\partial C_e}{\partial \phi}$$
$$= G_o[\frac{3}{2}a_3X^2 + \eta(p-2)X^{p-2}X_s^q \cos p\phi - \eta pX^{p-2}X_s^q \cos p\phi]$$
$$= G_o[\frac{3}{2}a_3X^2 - 2\eta X^{p-2}X_s^q \cos p\phi]$$
$$= G_o[\frac{3}{2}a_3X^2 + \frac{3}{2}a_3(X^2 - X_o^2)] > 0 \qquad (4.21)$$

Since $a_3 > 0$, this inequality is satisfied if

$$X^2 > \frac{1}{2} X_o^2 \qquad (4.22)$$

For forced oscillations ($X_o^2 = -1$), Ineq. 4.22 is satisfied automatically.

The second part of Ineq. 4.20 is

$$\frac{\partial C_e}{\partial \phi} \frac{\partial G_e}{\partial X} + \frac{\partial C_e}{\partial X} \frac{\partial G_e}{\partial \phi}$$
$$= -\frac{G_o^2}{\omega X}\{[\eta pX^{p-2}X_s^q \cos p\phi][\frac{3}{2}a_3X^2 + \eta(p-2)X^{p-2}X_s^q \cos p\phi]$$
$$+ [\eta pX^{p-2}X_s^q \sin p\phi][\eta(p-2)X^{p-2}X_s^q \sin p\phi]\}$$
$$= -\frac{G_o^2}{\omega} pX^{p-1}X_s^q[\frac{3}{2}a_3\eta \cos p\phi - \eta^2(p-2)X^{p-4}X_s^q] > 0 \qquad (4.23)$$

This inequality is satisfied if

$$\eta \cos p\phi < -\frac{2}{3}\eta^2 \frac{p-2}{a_3} X^{p-4}X_s^q \qquad (4.24)$$

For Ineq. 4.24 three different cases of subharmonic synchronization have to be distinguished: $p = 1$, $p = 2$ and $p > 2$.

a. $p = 1$

If $p = 1$, then the right-hand side of Ineq. 4.24 is positive. The range of the phase angle ϕ for which the oscillation is stable is therefore larger than π. For positive η it extends

approximately from $\pi/2$ to $3\pi/2$ and for negative η from $-\pi/2$ to $\pi/2$ (Fig. 4.1). Equation 4.17 shows that for $X_o^2 = -1$ (forced oscillation) the possible variation of ϕ for which X is real is smaller than π. Inequality 4.24 is therefore automatically satisfied. For $X_o^2 = 1$ (free oscillations), ϕ may however vary over a much wider range than π as indicated in Fig. 4.1, and therefore Ineq. 4.24 must be considered. The critical regions are shaded in Fig. 4.1. In these regions $\eta \cos \phi$ is positive. Using Eq. 4.17, Ineq. 4.24 can be simplified to:

$$\cos^2 \phi < \frac{1}{2} \left(\frac{X_o^2}{X^2} - 1\right) \tag{4.25}$$

Fig. 4.2 shows the region of possible steady-state oscillations defined by Ineqs. 4.22 and 4.25 for $\eta > 0$. The regions for $\eta < 0$ can easily be obtained by a phase shift of 180°.

b. $p = 2$

For $p = 2$, Ineq. 4.24 reduces to

$$\eta \cos 2\phi < 0 \tag{4.26}$$

The total variation of (2ϕ) is therefore π for $X_o^2 = 1$ and smaller for $X_o^2 = -1$ (Fig. 4.1).

c. $p > 2$

If $p > 2$, then the right-hand side of Ineq. 4.24 is negative. Therefore the variation of the phase angle for which the oscillations are stable is smaller than π (Fig. 4.1). Using Eq. 4.17, Ineq. 4.24 transforms to

$$\cos^2 p\phi > \frac{p-2}{2} \left(1 - \frac{X_o^2}{X^2}\right) \tag{4.27}$$

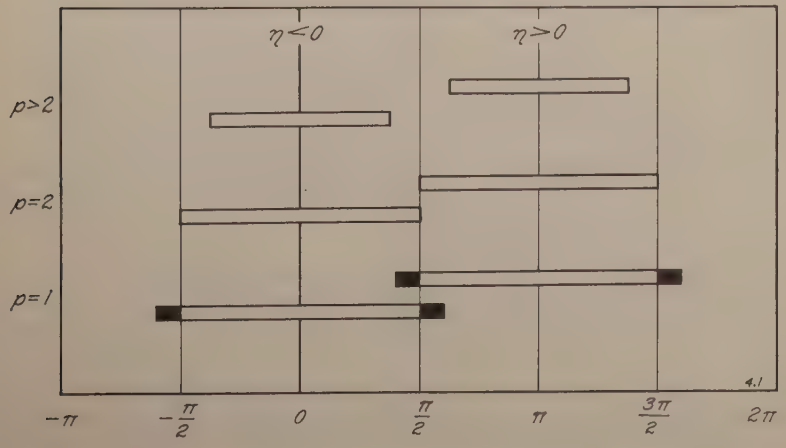

Fig. 4.1. Range of $p\phi$ for Stable Synchronous Oscillations ($X_o^2 = 1$)

Equation 4.17 shows that for this variation and for $X_o^2 = 1$, $X > 1$ and therefore Ineq. 4.22 need not be considered. The region of possible steady-state oscillations is indicated in Fig. 4.3 for $p = 3$.

For $X_o^2 = -1$, Ineq. 4.27 can be satisfied for $p = 3$ only (Fig. 4.3b). For $p \geq 4$, Ineq. 4.27 cannot possibly be satisfied. Therefore, for a first-order approximation no forced subharmonic oscillations are possible for $p \geq 4$. This, of course, would not be true if more than the first two terms of Eq. 4.2 were considered.

8. Particular Examples of Subharmonic Synchronization

The discussion in this section deals with three examples of subharmonic synchronization:

$$\omega = 3\omega_s, \quad 2\omega = \omega_s, \quad 3\omega = \omega_s.$$

a. $\omega = 3\omega_s$ $(p = 1, q = 3)$

For this example η from Table I is $\frac{1}{4}a_3$. Eq. 4.17 is then

$$X^2 = X_o^2 - \frac{1}{3}\frac{X_s^3}{X}\cos\phi$$

The regions for stability for $X_o^2 = 1$ are indicated in Fig. 4.2a. For $X_o^2 = -1$, all steady-state oscillations are stable as shown in Fig. 4.2b. Fig. 4.2 also shows the variation of X with $p\phi$ for X_s = constant. The variations of the phase angle are

$$-\pi/2 < \Delta\phi < \pi/2 \qquad (X_o^2 = 1)$$

$$0 < \Delta\phi < \pi \qquad (X_o^2 = -1)$$

b. $2\omega = \omega_s$ $(p = 2, q = 1)$

From Table I η is a_2 and Eq. 4.17 is

$$X^2 = X_o^2 - \frac{4}{3}\frac{a_2}{a_3}X_s\cos 2\phi$$

As shown in Fig. 4.1, the range of 2ϕ is π for $X_o^2 = 1$. For $X_o^2 = -1$, all steady-state oscillations are stable.

c. $3\omega = \omega_s$ $(p = 3, q = 1)$

For this case η is $\frac{3}{4}a_3$. Eq. 4.17 is then

$$X^2 = X_o^2 - X X_s \cos 3\phi$$

For $X_o^2 = 1$ the region of stability is shown in the shaded area of Fig. 4.3a; for $X_o^2 = -1$, in Fig. 4.3b. The variation of X

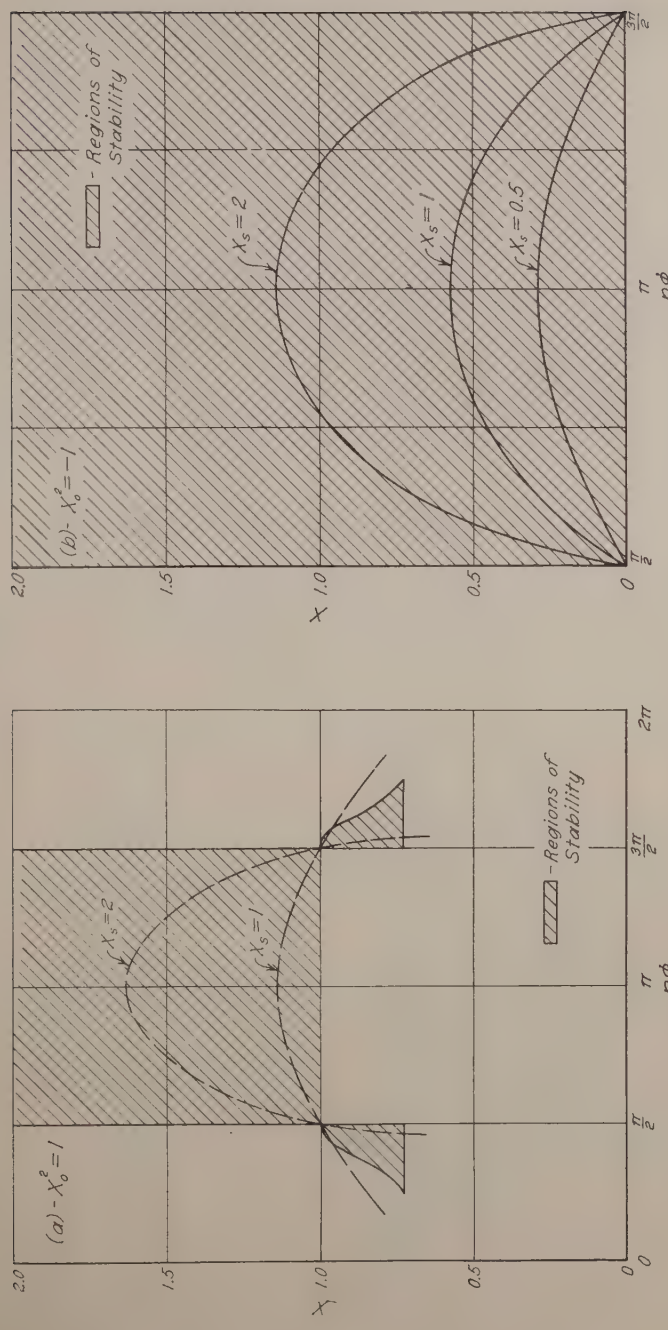

Fig. 4.2. Regions of Stability and Variation of X as a Function of $p\phi$ for $p = 3$, $q = 1$

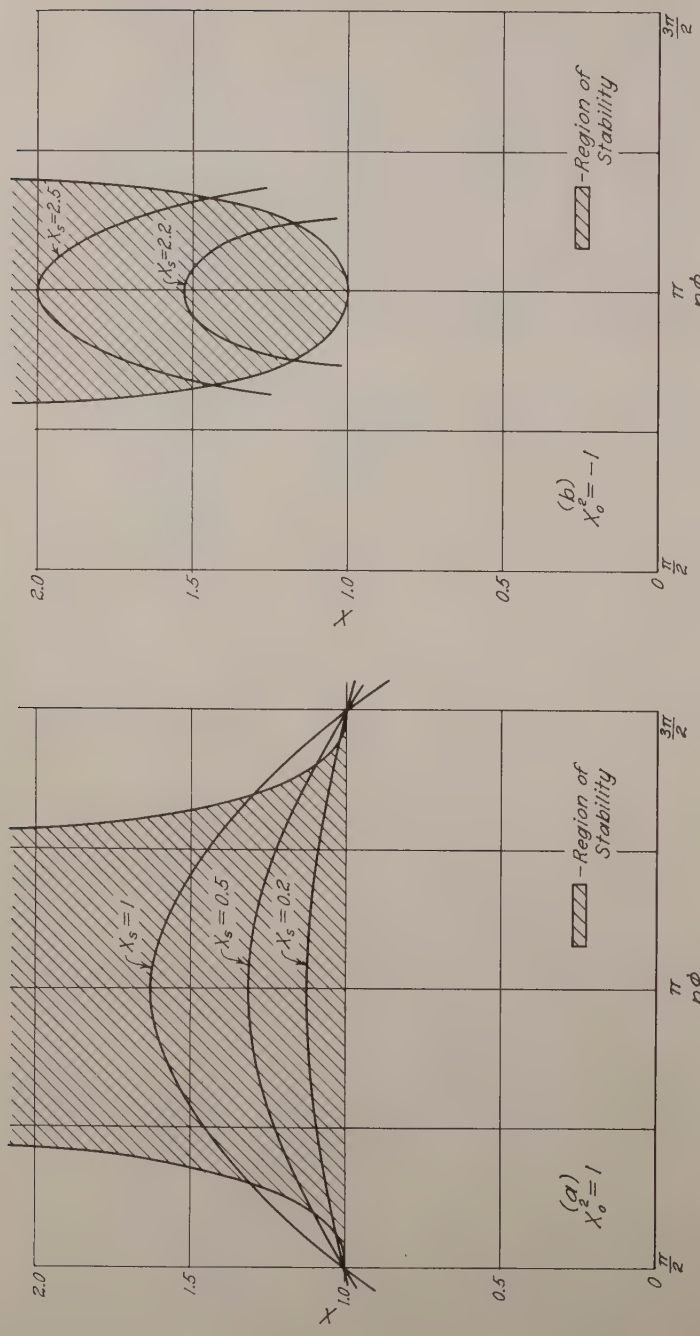

Fig. 4.3. Region of Stability and Variation of χ as a Function of $p\phi$ for $p = 3$, $q = 1$

with $p\phi$ is also shown in Fig. 4.3. As can be seen from these figures, the total change in phase is

$$\pi/2 < \Delta 3\phi < \pi \qquad (X_o^2 = 1)$$

$$0 < \Delta 3\phi < \pi/2 \qquad (X_o^2 = -1)$$

9. The Bandwidth of Synchronization

So far in this bulletin the limits of stable synchronization have been expressed in terms of phase. For practical purposes it is, however, more important to know the maximum permissible deviation in frequency, or the bandwidth of synchronization $(\Delta\omega/\omega)_o$. It is of course possible to find precise expressions, but in most cases an approximate value is satisfactory. Such an approximate value can be found if it is assumed that $X = 1$ and that the phase angle varies by an amount π either from $-\pi/2$ to $\pi/2$ or from $\pi/2$ to $3\pi/2$. The bandwidth of synchronization is then (from Eq. 4.14)

$$\left(\frac{\Delta\omega}{\omega}\right)_o = \eta \frac{G_o}{\omega C} X_s^q = \frac{a_\lambda G_o}{2^{\lambda-1}\omega C} \binom{\lambda}{q} X_s^q \qquad (4.28)$$

This equation shows that the bandwidth of synchronization becomes rapidly smaller as $\lambda = p+q-1$ increases.

V. EXPERIMENTAL VERIFICATION

In the discussion of the theory some questionable approximations have been made. It was therefore necessary to investigate how well the theoretical results are confirmed by the experiment. The circuit used in the experiment is shown in Fig. 5.1.

In this circuit, designed by Dr. Giuseppe Francini, a conventional transitron circuit served as the nonlinear element. The operating point of the tube was adjusted by the potentiometers R_1 and R_2 and slight changes in the current-voltage characteristic were made by changing tubes. The synchronizing voltage was supplied by a Hewlett-Packard oscillator. The experimental results were obtained using resonant circuits tuned to a frequency of 6 kc and having a Q of about 35.

Synchronization was observed with the aid of Lissajou figures on the screen of an oscilloscope. The voltages V and V_s were measured directly across the oscillatory circuit and R_s.

It was to be expected that the critical part of the theory was that dealing with stability rather than equilibrium. The conditions of stability for cases ($3\omega = \omega_s$) and ($\omega = 3\omega_s$) were checked in this experiment. The results of the experiment are shown in Figs. 5.2 and 5.3. It can be seen from these results that the correspondence between theory and experiment is reasonably good.

Equation 3.28 was also verified by experiment. Good correspondence was obtained for ratios of V_s/V as high as 0.15. The experimental results for $V_s/V = 0.1$ are shown in Fig. 5.4. Generally, the maximum variation of the phase angle was somewhat larger than π; therefore, the measured values of $(\Delta\omega/\omega)_o$ are somewhat too large.

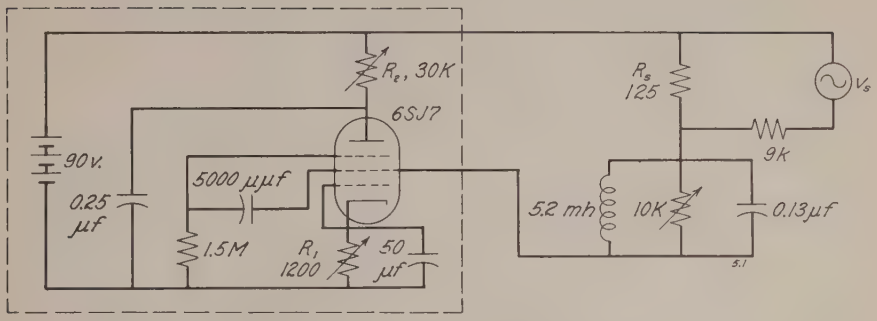

Fig. 5.1. Circuit Used for Experimental Verification of the Theoretical Results

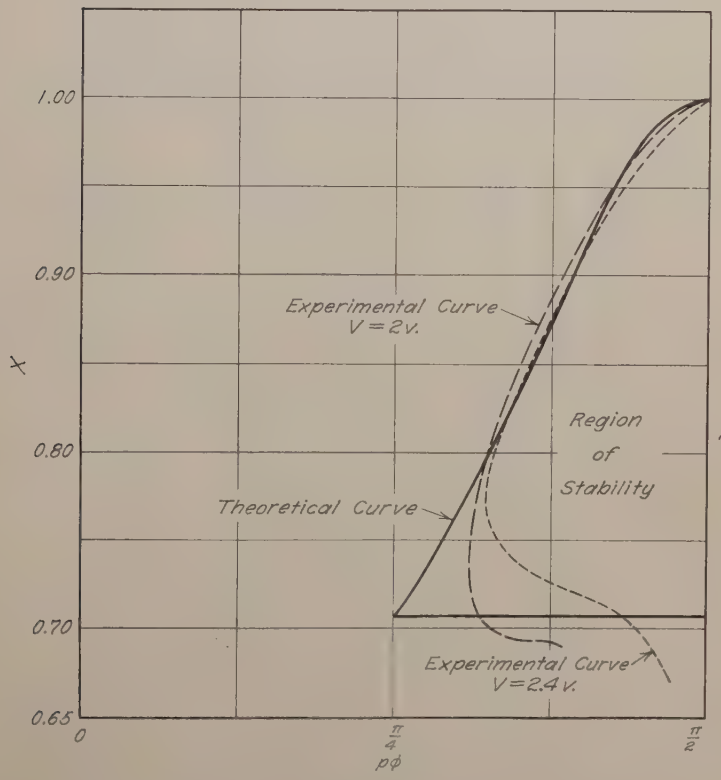

Fig. 5.2. Region of Stability for $p = 1$, $q = 3$ ($X_o^2 = 1$)

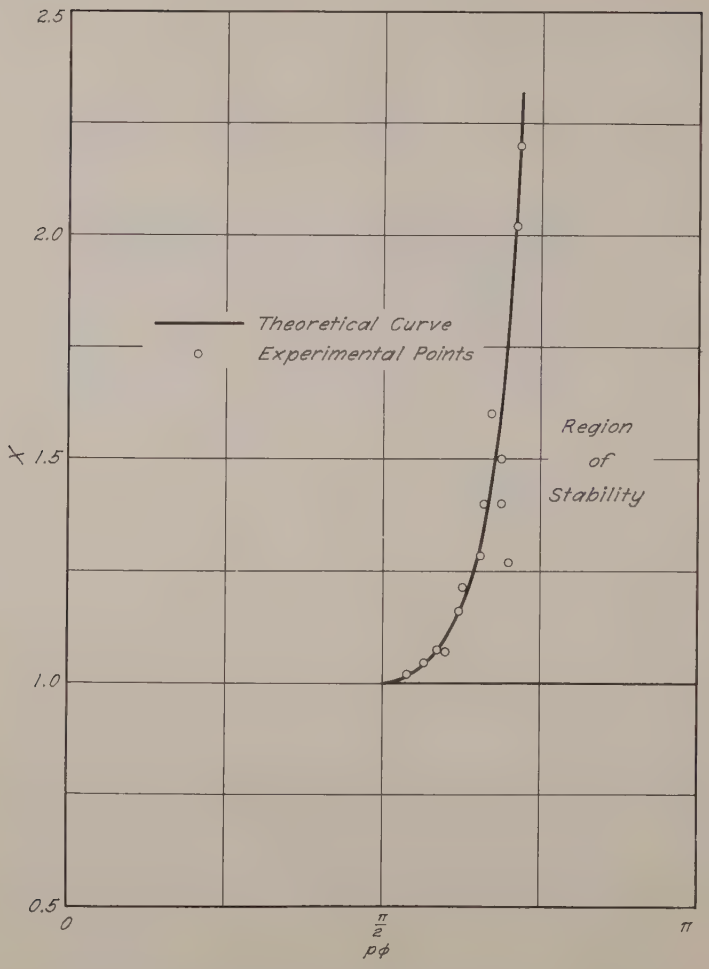

Fig. 5.3. Region of Stability for $p = 3$, $q = 1$ ($X_o^2 = 1$)

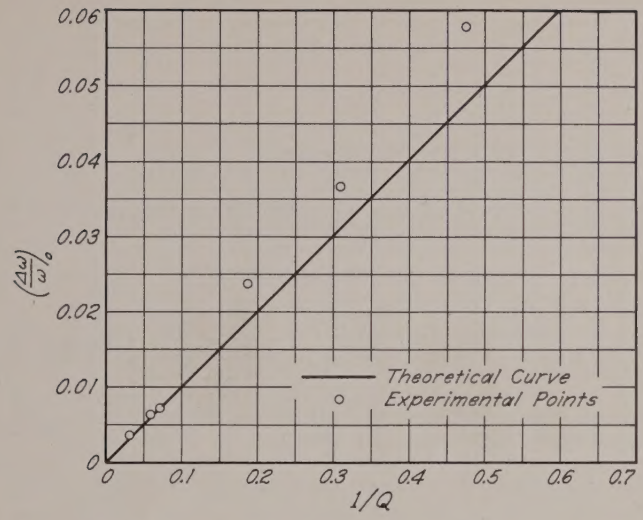

Fig. 5.4. Bandwidth of Synchronization, $(\Delta\omega/\omega)_o$, as a Function of Q, $G/\omega C$, for External 1:1 Synchronization

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